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SOME FUNCTION SPACE TOPOLOGIES

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF ALBERTA

EDMONTON, ALBERTA

SPRING, 1969

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "SOME FUNCTION SPACE TOPOLOGIES", submitted by PETER BANCROFT in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis introduces a function space topology in which the set of continuous functions is closed for any topological domain and range spaces. This topology and several other function space topologies are then compared in terms of containment. Finally some separation properties of the various topologies are found.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor, Dr. S.A. Naimpally, for suggesting this topic and for his prompt and helpful assistance whenever it was required.

I would also like to thank my father, Philip Bancroft, Jr., for providing the financial support for this research.

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INTRODUCTION

In the past several years many new function space topologies have been introduced. Among these is the topology of uniform convergence, which has the property that the set of continuous functions is closed. However, this topology is only defined when the range space of the functions is a uniform space. Another function space topology, the near topology, maintains the property of the continuous functions being closed when the range space of the functions is regular. This paper introduces a new function space topology, the τ topology, in which the set of continuous functions is closed when the domain and range spaces are any topological spaces.

The τ topology and other function space topologies are then compared, the results being as follows:

	c	1 Pc	2 K	3 σ	4 Γ_1	5 Γ_2	6 Γ_3	7 Γ_4	8 uc	9 N	10 T	11 τ
1	Pc	0	0	1	1	1	1	1	0	1'	0	0
2	K	#	0	3	2	2	2	2	c	Q2	cL	0
3	σ	#	c	c3	0	0	0	0	cC	Q	cCL	0
4	Γ_1	#	c	c3	0	0	0	0	cC	Q	cCL	0
5	Γ_2	#	cC2	cC2	cC2	0	0	0	cC2	c2'C	c2CL	cC2 or cC2'
6	Γ_3	#	cC2	cC2	cC2	1	0	0	cC2	c12'C	c2CL	cC2 or cC12'
7	Γ_4	#	cC2	cC2	cC2	cC1	cC	0	cC2	c12'C	c2CL	cC2 or cC12'
8	uc	#	cC	cC2	cC2	cC2	cC2	cC2	0	c22'C	cCL	cC
9	N	#	cC2	cC2	cC2	C	C	C	cC2	0	cC2L	cC2
10	T	#	#	#	#	#	#	#	#	#	0	0
11	τ	#	#	#	#	#	#	#	#	#	#	0

Assumptions:

0	none
1	X is T_1
1'	Y is T_1
2	X is T_2
2'	Y is T_2
3	X is T_3
c	X is compact
L	X is locally connected
C	on the continuous functions
Q	on the closed functions
#	counterexample

The notation used in the list of topologies and assumptions will be described later. The chart is read as follows: the intersection of a row and a column gives the conditions under which the topology of the row is contained in the topology of the column. Also, the numbers of the row and column give the number of the theorem where the result is found. For example, the K topology is contained in the T topology on the continuous functions when X is locally connected. This result is Theorem 2.10 of Chapter 2.

While the conditions under which comparison of topologies is possible usually involve the domain space of the functions, the separation properties of the function spaces usually are related to the

range space of the functions. This is seen in Chapter 3 where the separation properties of various function spaces are discussed.

CHAPTER I

Notation, Definitions, the τ Topology

The following notation will be used throughout this paper.

C	The set of continuous functions from X to Y .
C_o	The set of constant functions from X to Y .
F	The set of all functions from X to Y .
$G(f)$	The graph of f , i.e. $\{(x,f(x)) : x \varepsilon X\}$.
Q	The set of closed functions from X to Y , i.e. $\{f :$ if A is closed in X , $f(A)$ is closed in $Y\}$.
X	The domain space of the function or functions.
Y	The range space of the function or functions.
y^*	The constant function $y^*(X) = y$.
$\partial^*(U)$	$\{y^* : y \varepsilon \partial U\}$.
X	The set of open subsets of X .
Y	The set of open subsets of Y .
$\tau(P,U)$	$\{f \varepsilon F : f^{-1}(U) = P , U \varepsilon Y\}$.
(f,U)	$\tau(f^{-1}(U),U)$.
(A,U)	$\{f \varepsilon F : f(A) \subset U , U \varepsilon Y\}$.
$W(f,\alpha)$	$\{g \varepsilon F : g(x) \varepsilon \alpha^*(f(x))$ for all $x \varepsilon X\}$.
$\alpha^*(A)$	$\cup\{U \varepsilon \alpha : A \cap U \neq \phi\}$.

The topologies on F to be discussed are denoted and defined as follows.

pc — The pointwise convergence topology, generated by the subbase $\{(A,U) : A$ is a point of $X\}$.

K — The compact-open topology, generated by the subbase $\{(A,U) : A$ is compact in $X\}$.

σ — Any sigma topology. Let σ be an open cover of X , and let $\{A_i : i \in I\}$ be the collection of closed subsets of X such that each A_i is in an element of σ . Then $\{(A_i,U) : i \in I\}$ is a subbasis for the topology.

$\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ — Graph topologies. A collection $\underset{\sim}{U}$ of open subsets of $X \times Y$ is called a <u>clover</u> of $G(f)$, $f \in F$, iff $G(f) \cap U \neq \phi$ for each $U \in \underset{\sim}{U}$ and $\underset{\sim}{U}$ covers $G(f)$ If $\underset{\sim}{U}$ is a clover of $G(f)$ then the set of functions whose graphs have $\underset{\sim}{U}$ as a clover is a basic open neighborhood of f in the graph topology corresponding to the clovers of which $\underset{\sim}{U}$ is a member.

Γ_1 — The family of clovers consists of sets of the form $\bigcap_{i=1}^{n} [X \times Y - A_i \times B_i]$ where A_i , B_i are closed subsets of X and Y respectively. This topology is the same as that with the subbasis $\{(A,U) : A$ closed in $X\}$.

Γ_2 — Each clover of the family consists of one open set in $X \times Y$.

Γ_3 — Each clover of the family consists of a finite number of open sets in $X \times Y$.

Γ_4 — Each clover of the family is locally finite.

uc The topology of uniform convergence. If Y is a uniform space, with uniformity V, then let $W(Y) = \{(f,g) : f , g \in F , (f(x) , g(x)) \in V$ for each $x \in X\}$. Then $\{W(v) : v \in V\}$ forms a base for a uniformity for F, and the topology of this uniformity is the topology of uniform convergence.

N The near topology. Let α be an open cover of Y. Let $\alpha^*(A) = \cup\{U \in \alpha : A \cap U \neq \phi\}$. Then for $f \in F$, a subbasis for the near topology is sets of the form $W(f,\alpha) = \{g \in F : g(x) \in \alpha^*(f(x))$ for all $x \in X\}$.

T The connected open topology. For each connected subset K of X and each $U , V \in (Y)$ a subbasis for T is sets of the form $W(K;U,V) = \{f \in F : f(K) \subset U \cup V , f(K) \cap U \neq \phi \neq f(K) \cap V\}$.

τ The topology generated by the subbasis $\tau(P,U) = \{f \in F : f^{-1}(U) = P , U \in (Y)\}$ where P is a non empty subset of X.

<u>Theorem</u>: The set of continuous functions is closed in (F,τ).

<u>Proof</u>:

If $f \in F$ is not continuous, there is $U \in (Y)$ such that $f^{-1}(U) \notin (X)$. Then (f,U) is an open set containing f but no continuous functions. Thus the complement of C is open, so C is closed.

CHAPTER II

Comparison of Topologies

A topology τ_1 is said to be contained in $(\subset)$ a topology τ_2 if every open set in τ_1 is open in τ_2. Theorems corresponding to positions 1.1 , 2.2 , etc. of the chart in the introduction are trivial and have been omitted.

<u>Theorem 1.2</u> The pointwise convergence topology is contained in the compact open topology.

<u>Proof</u>:

The pointwise convergence topology is generated by sets of the form (A,U) where A is a point in X. Since a point is a compact set, (A,U) is one of the subbasic sets of the compact-open topology. Thus $pc \subset K$.

<u>Theorem 1.3</u> The pointwise convergence topology is contained in any σ topology when X is T_1.

<u>Proof</u>:

The pointwise convergence topology is generated by sets of the form (A,U) where A is a point of X. Since X is T_1, A is closed, and since the σ topology is defined from a cover of X, A

is in some element of that cover. Thus (A,U) is one of the defining open sets of the σ topology. Thus $pc \subset \sigma$ when X is T_1 .

Theorem 1.4 The pointwise convergence topology is contained in the Γ_1 topology when X is T_1 .

Proof:

The pointwise convergence topology is generated by sets of the form (A,U) where A is a point of X . If X is T_1 , A is closed in X , and (A,U) is one of the subbasic sets generating Γ_1 . Thus $pc \subset \Gamma_1$ when X is T_1 .

Theorem 1.5 The pointwise convergence topology is contained in the Γ_2 topology when X is T_1 .

Proof:

The pointwise convergence topology is contained in the Γ_1 topology when X is T_1 by Theorem 1.4 . The Γ_1 topology is contained in the Γ_2 topology generally by Theorem 4.5 . Thus $pc \subset \Gamma_2$ when X is T_1 .

Theorem 1.6 The pointwise convergence topology is contained in the Γ_3 topology when X is T_1 .

Proof:

The pointwise convergence topology is contained in the Γ_1 topology when X is T_1 by Theorem 1.4 . The Γ_1 topology is contained in the Γ_3 topology in general by Theorem 4.6 . Thus $pc \subset \Gamma_3$ when X is T_1 .

Theorem 1.7 The pointwise convergence topology is contained in the Γ_4 topology when X is T_1 .

Proof:

The pointwise convergence topology is contained in the Γ_1 topology when X is T_1 by Theorem 1.4 . The Γ_1 topology is contained in the Γ_4 topology in general by Theorem 4.7 . Thus $pc \subset \Gamma_4$ when X is T_1 .

Theorem 1.8 The pointwise convergence topology is contained in the uniform convergence topology.

Proof:

To have the uniform convergence topology on F , Y must be a uniform space. Call its uniformity U . The pointwise convergence topology is the topology of the uniformity with subbase sets of the form $\{(f,g) : f , g \varepsilon F , (f(x) , g(x)) \varepsilon V\}$ for $V \varepsilon U$ and $x \varepsilon X$. Since $\{(f,g) : (f(x) , g(x)) \varepsilon V$ for each $x \varepsilon X\} \subset \{(f,g) : (f(x_o) , g(x_o)) \varepsilon V\}$ for some given $x_o \varepsilon X$, each basic set for the uniformity giving the

topology of uniform convergence is contained in a subbasic set for the uniformity giving the pointwise convergence topology. Thus the pointwise convergence topology is contained in the uniform convergence topology.

Theorem 1.9 The pointwise convergence topology is contained in the near topology if Y is T_1 .

Proof:

For any subbasic set (x,U) in the pointwise convergence topology, let $f \varepsilon (x,U)$. Then $\alpha = \{U , Y - f(x)\}$ is an open cover of Y so $W(f,\alpha)$ is an open set in the near topology and $W(f,\alpha) \subset (x,U)$ so (x,U) is open in the near topology. Thus $pc \subset N$ if Y is T_1 .

Theorem 1.10 The pointwise convergence topology is contained in the connected open topology.

Proof:

The subbasic set (x,U) of the pointwise convergence topology is equal to the subbasic set $W(x;U,U)$ of the connected open topology so $pc \subset T$.

Theorem 1.11 The pointwise convergence topology is contained in the τ topology.

Proof:

For the subbasic set (x,U) in the pointwise convergence topology let $f \in (x,U)$. Then (f,U) is an open set in the τ topology and $(f,U) \subset (x,U)$ so (x,U) is open in the τ topology. Thus $pc \subset \tau$.

Theorem 2.1 The compact open topology is not contained in the pointwise convergence topology on C when $X = Y = [0,1]$.

Proof: Counterexample

Let $X = Y = [0,1]$ with the usual topology and let $f = 0^*$, the constant 0 function. Consider the K-open set $([0,1],[0,\frac{1}{2}))$. For any basic pc open set $\bigcap_{i=1}^{n} (x_i,U_i)$ containing f , there is a continuous function g in $\bigcap_{i=1}^{n} (x_i,U_i)$ and not in $([0,1],[0,\frac{1}{2}))$. Since $[0,1]$ is infinite, there is an $x \in [0,1]$ such that $x \notin \bigcup_{i=1}^{n} x_i$. $\bigcup_{i=1}^{n} x_i$ is closed in $[0,1]$ and $[0,1]$ is completely regular so there is a continuous function g such that $g(x) = 1$, $g(\bigcup_{i=1}^{n} x_i) = 0$. This g is the required function, so $([0,1],[0,\frac{1}{2}))$ is not open in the pc topology.

Theorem 2.3 The compact open topology is contained in every σ topology when X is Γ_3 .

Proof:

Let (A,U) be a subbasic open set in the compact open

topology. The σ topology is defined form an open cover σ of X , so σ is an open cover of A . Thus for $p \varepsilon A$ there is an $S_p \varepsilon \sigma$ such that $p \varepsilon S_p$. Since X is T_3 , p has an open neighborhood V_p such that $\overline{V}_p \subset S_p$. The V_p is for all $p \varepsilon A$ form an open cover of A so, since A is compact, there is a finite subcover $V_{p_i},\ldots,V_{p_n}$ of A . Also A is closed since it is a compact subset of a hausdorff space. Thus $\overline{V}_{p_i} \cap A = B_i$ is closed and contained in S_{p_i} for $1 \leq i \leq n$. So (B_i,U) is open in the σ topology. Now $f \varepsilon (A,U)$ implies $f \varepsilon (B_i,U)$ since $B_i \subset A$. So $(A,U) \subset \bigcap_{i=1}^{n} (B_i,U)$. Also if $g \varepsilon \bigcap_{i=1}^{n} (B_i,U)$ then $g \varepsilon (A,U)$ since $A = \bigcup_{i=1}^{n} B_i$. Thus $\bigcap_{i=1}^{n} (B_i,U) = (A,U)$ so (A,U) is open in the σ topology. Thus $K \subset \sigma$ when X is T_3 .

<u>Theorem 2.4</u> The compact open topology is contained in the Γ_1 topology when X is T_2 .

<u>Proof:</u>

Let (A,U) be a subbasic open set in the compact open topology. Since X is T_2 , A being compact is closed. Thus (A,U) is one of the generating open sets in the Γ_1 topology.

<u>Theorem 2.5</u> The compact open topology is contained in the Γ_2 topology when X is T_2 .

<u>Proof:</u>

The compact open topology is contained in the Γ_1

topology when X is T_2 by the Theorem 2.4 . The Γ_1 topology is contained in the Γ_2 topology in general by Theorem 4.5 . Thus $K \subset \Gamma_2$ when X is T_2 .

<u>Theorem 2.6</u> The compact open topology is contained in the Γ_3 topology when X is T_2 .

<u>Proof</u>:

The compact open topology is contained in the Γ_1 topology when X is T_2 by Theorem 2.4 . The Γ_1 topology is contained in the Γ_3 topology in general by Theorem 4.6 . Thus $K \subset \Gamma_3$ when X is T_2 .

<u>Theorem 2.7</u> The compact open topology is contained in the Γ_4 topology when X is T_2 .

<u>Proof</u>:

The compact open topology is contained in the Γ_1 topology when X is T_2 by Theorem 2.4 . The Γ_1 topology is contained in the Γ_4 topology in general by Theorem 4.7 . Thus $K \subset \Gamma_4$ when X is T_2.

<u>Theorem 2.8</u> The compact open topology is contained in the uniform convergence topology on the set of continuous functions.

<u>Proof</u>:

Let (A,U) be a subbasic open set in the compact open

set in the compact open topology. Let $f \in (A,U) \cap C$. Then $f(A)$ is compact, so if Y has uniformity V , there is a $v \in V$ such that $v[f(A)] \subset U$ (Theorem 33, Page 119, Kelley [4]). Now if $g \in W(v)[f]$, then $(f(x), g(x)) \in v$ for all $x \in X$. Thus $g(A) \subset v[f(A)] \subset U$. So $g \in (A,U)$, and $W(v)[f]$ is a uc-open set containing f and contained in (A,U) so (A,U) is open in the uniform convergence topology.

<u>Theorem 2.9</u> The compact open topology is contained in the near topology on the set of closed functions when X is T_2 .

<u>Proof</u>:

The compact open topology is contained in the Γ_1 topology when X is T_2 by Theorem 2.4 . The Γ_1 topology is contained in the near topology on the set of closed functions by Theorem 4.9 . Thus $K \subset N$ on Q when X is T_2 .

<u>Theorem 2.10</u> The compact open topology is contained in the connected open topology on the set of continuous functions when X is locally connected.

<u>Proof</u>:

Let (A,U) be a subbasic open set in the compact open topology and let $f \in (A,U)$, f continuous. Then $A \subset f^{-1}(U)$ which is open in X . Since X is locally connected each point of A has a

connected, open neighborhood contained in U . Since A is compact, this open cover of A had a finite subcover $V_1,\dots,V_n$. Then $\bigcap_{i=1}^{n} W(V_i,U,U)$ is an open T-neighborhood of f and is contained in (A,U) since $A \subset \bigcup_{i=1}^{n} V_i$. Thus (A,U) is open in the connected open topology.

Theorem 2.11 The compact open topology is contained in the τ topology.

Proof:

Let (A,U) be a subbasic open set in the compact open topology. Let $f \in (A,U)$. Then (f,U) is an open set in the topology containing f and contained in (A,U) since $A \subset f^{-1}(U)$. Thus (A,U) is open in the τ topology.

Theorem 3.1 No σ topology is contained in the pointwise convergence topology on C when $X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in every σ topology when X is T_3 by Theorem 2.3 . The compact open topology is not contained in the pointwise convergence topology on C when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus no σ topology can be contained in the pointwise convergence topology on C when $X = Y = [0,1]$ with the usual topology.

Theorem 3.2 Every σ topology is contained in the compact open topology when X is compact.

Proof:

Let (A,U) be a subbasic open set in a σ topology. Then A is closed, and so compact since X is compact. Thus (A,U) is an open set in the compact open topology.

Theorem 3.4 Every σ topology is contained in the Γ_1 topology.

Proof:

Let (A,U) be a subbasic open set in a σ topology. Then A is closed, so (A,U) is open in the Γ_1 topology.

Theorem 3.5 Every σ topology is contained in the Γ_2 topology.

Proof:

Every σ topology is contained in the Γ_1 topology by Theorem 3.4 . The Γ_1 topology is contained in the Γ_2 topology by Theorem 4.5 . Thus $\sigma \subset \Gamma_2$.

Theorem 3.6 Every σ topology is contained in the Γ_3 topology.

Proof:

Every σ topology is contained in the Γ_1 topology by Theorem 3.4 . The Γ_1 topology is contained in the Γ_3 topology by Theorem 4.6 . Thus $\sigma \subset \Gamma_3$.

Theorem 3.7 Every σ topology is contained in the Γ_4 topology.

Proof:

Every σ topology is contained in the Γ_1 topology by Theorem 3.4 . The Γ_1 topology is contained in the Γ_4 topology by Theorem 4.7 . Thus $\sigma \subset \Gamma_4$.

Theorem 3.8 Every σ topology is contained in the uniform convergence topology on the set of continuous functions when X is compact.

Proof:

Every σ topology is contained in the compact open topology when X is compact by Theorem 3.2 . The compact open topology is contained in the uniform convergence topology on the set of continuous functions by Theorem 2.8 . Thus $\sigma \subset uc$ on C when X is compact.

Theorem 3.9 Every σ topology is contained in the near topology on the set of closed functions.

Proof:

Every σ topology is contained in the Γ_1 topology by Theorem 3.4 . The Γ_1 topology is contained in the near topology on the set of closed functions by Theorem 4.9 . Thus $\sigma \subset N$ on Q .

Theorem 3.10 Every σ topology is contained in the connected open topology on the set of continuous functions when X is compact and locally connected.

Proof:

Every σ topology is contained in the compact open topology when X is compact by Theorem 3.2 . The compact open topology is contained in the connected open topology on the set of continuous functions when X is locally connected by Theorem 2.10 . Thus $\sigma \subset T$ on C when X is locally connected and compact.

Theorem 3.11 Every σ topology is contained in the τ topology.

Proof:

Let (A,U) be a subbasic open set in a σ topology, and let $f \in (A,U)$. Then (f,U) is a τ-open neighborhood of f contained in (A,U) , so (A,U) is open in the τ topology.

Theorem 4.1 The Γ_1 topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in the Γ_1 topology when X is T_2 by Theorem 2.4 . The compact open topology is not contained in the pointwise convergence topology on the continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus the Γ_1 topology is not contained in the pointwise convergence topology on C when $X = Y = [0,1]$.

Theorem 4.2 The Γ_1 topology is contained in the compact open topology when X is compact.

Proof:

Let (A,U) be a subbasic open set in the Γ_1 topology. A is closed, so is compact since X is compact. Thus (A,U) is open in the compact open topology.

Theorem 4.3 The Γ_1 topology is contained in any σ topology when X is compact and T_2 .

Proof:

The Γ_1 topology is contained in the compact open topology when X is compact by Theorem 4.2 . The compact open topology is contained in any σ topology when X is T_3 by Theorem 2.3 . Since X being compact T_2 implies X is T_3 , the Γ_1 topology is contained in any σ topology when X is compact T_2 .

<u>Theorem 4.5</u> The Γ_1 topology is contained in the Γ_2 topology.

<u>Proof</u>:

Let (A,U) be a subbasic open set in the Γ_1 topology. Then $f \varepsilon (A,U)$ means $G(f) \subset [(XxU) \cup (A^c xY)]$ which is open in XxY . This set is the required clover for the Γ_2 topology. Thus $\Gamma_1 \subset \Gamma_2$.

<u>Theorem 4.6</u> The Γ_1 topology is contained in the Γ_3 topology.

<u>Proof</u>:

The Γ_1 topology is contained in the Γ_2 topology by the Theorem 4.5 . The Γ_2 topology is contained in the Γ_3 topology by Theorem 5.6 . Thus $\Gamma_1 \subset \Gamma_3$.

<u>Theorem 4.7</u> The Γ_1 topology is contained in the Γ_4 topology.

<u>Proof</u>:

The Γ_1 topology is contained in the Γ_2 topology by Theorem 4.5 . The Γ_2 topology is contained in the Γ_4 topology by Theorem 5.7 . Thus $\Gamma_1 \subset \Gamma_4$.

<u>Theorem 4.8</u> The Γ_1 topology is contained in the uniform convergence topology on the set of continuous functions when X is compact.

Proof:

The Γ_1 topology is contained in the compact open topology when X is compact by Theorem 4.2 . The compact open topology is contained in the uniform convergence topology on the set of continuous functions by Theorem 2.8 . Thus $\Gamma_1 \subset uc$ on C when X is compact.

Theorem 4.9 The Γ_1 topology is contained in the near topology on the set of closed functions.

Proof:

Let (A,U) be a subbasic open set in the Γ_1 topology, and let $f \in (A,U)$, f a closed function. Then f(A) is closed since A is closed, so let $\alpha = \{U, Y - f(A)\}$. Then α is an open cover of Y , and $w(f,\alpha)$ is an open set in the near topology containing f and contained in (A,U) so (A,U) is open in the near topology. Thus $\Gamma_1 \subset N$ on Q .

Theorem 4.10 The Γ_1 topology is contained in the connected open topology on the set of continuous functions when X is compact and locally connected.

Proof:

The Γ_1 topology is contained in the compact open topology when X is compact by Theorem 4.2 . The compact open topology is contained in the connected open topology on the set of continuous functions

when X is locally connected by Theorem 2.10 . Thus $\Gamma_1 \subset T$ on C when X is locally connected.

<u>Theorem 4.11</u> The Γ_1 topology is contained in the τ topology.

<u>Proof</u>:

Let (A,U) be a subbasic open set in the Γ_1 topology. Let $f \varepsilon (A,U)$. Then (f,U) is a τ-open set containing f and contained in (A,U) so (A,U) is open in the τ topology.

<u>Theorem 5.1</u> The Γ_2 topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

<u>Proof</u>:

The compact open topology is contained in the Γ_2 topology when X is T_2 by Theorem 2.5 . The compact open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1. Thus the Γ_2 topology is not contained in the pointwise convergence topology on C when $X = Y = [0,1]$ with the usual topology.

<u>Theorem 5.2</u> The Γ_2 topology is contained in the compact open topology on the set of continuous functions when X is compact and T_2 .

Proof:

Let M be open in XxY and let $G(f) \subset M$, f continuous. For each $(x, f(x)) \varepsilon G(f)$ there are open sets $G_x \subset X$ and $H_x \subset Y$ such that $x \varepsilon G_x$, $f(x) \varepsilon H_x$ and $G_x \times H_x \subset M$ since M is open. Since f is continuous there is an open neighborhood W_x of x such that $f(W_x) \subset H_x$. Let $U_x = G_x \cap W_x$. Since x is compact T_2 there is an open neighborhood V_x of x such that $\overline{V}_x \subset U_x$, so $f(\overline{V}_x) \subset H_x$. The collection $\{V_x : x \varepsilon X\}$ is an open cover of X so there is a finite subcover $V_{x_1}, \ldots, V_{x_n}$. Then $f \varepsilon \bigcap_{i=1}^{n} (\overline{V}_{x_i}, H_{x_i})$, $\overline{V}_{x_i}$ is compact since X is. Also, if $g \varepsilon \bigcap_{i=1}^{n} (\overline{V}_{x_i}, H_{x_i})$ and $(x , g(x)) \varepsilon G(g)$ then $x \varepsilon V_{x_i}$ for some i so $g(x) \varepsilon H_{x_i}$ so $(x , g(x)) \varepsilon \overline{V}_{x_i} x H_{x_i} \subset G_{k_i} x H_{x_i} \subset M$. Thus $G(g) \subset M$ so f and g are in the same Γ_2 open set $\underline{M}$. Thus $f \varepsilon \bigcap_{i=1}^{n} (\overline{V}_{x_i}, H_{x_i})$ $\underline{M}$ so $\underline{M}$ is open in the compact open topology on C .

Theorem 5.3 The Γ_2 topology is contained in any σ topology on the set of continuous functions when X is compact T_2 .

Proof:

The Γ_2 topology is contained in the compact open topology on the continuous functions when X is compact T_2 by Theorem 5.2 . The compact open topology is contained in any σ topology when X is T_3 by Theorem 2.3 . Since a compact T_2 space is T_3 , $\Gamma_2 \subset \sigma$ on C when X is compact T_2 .

Theorem 5.4 The Γ_2 topology is contained in the Γ_1 topology on the set of continuous functions when X is compact T_2 .

Proof:

The Γ_2 topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 by Theorem 5.2 . The compact open topology is contained in the Γ_1 topology when X is T_2 by Theorem 2.4 . Thus $\Gamma_2 \subset \Gamma_1$ on C when X is compact T_2 .

Theorem 5.6 The Γ_2 topology is contained in the Γ_3 topology.

Proof:

Since a single element clover of the graph of a function is finite, every basic Γ_2 neighborhood is a basic Γ_3 neighborhood, so $\Gamma_2 \subset \Gamma_3$.

Theorem 5.7 The Γ_2 topology is contained in the Γ_4 topology.

Proof:

The Γ_2 topology is contained in the Γ_3 topology by Theorem 5.6 . The Γ_3 topology is contained in the Γ_4 topology by Theorem 6.7 . Thus $\Gamma_2 \subset \Gamma_4$.

Theorem 5.8 The Γ_2 topology is contained in the uniform convergence topology on the set of continuous functions when X is compact T_2 .

Proof:

The Γ_2 topology is contained in the compact open topology

on the set of continuous functions when X is compact T_2 by Theorem 5.2 . The compact open topology is contained in the uniform convergence topology on the set of continuous functions by Theorem 2.8 . Thus $\Gamma_2 \subset uc$ on C when X is compact T_2 .

<u>Lemma 5.9</u> If X is compact, Y is T_2 , M is open in XxY and f is a continuous function and $G(f) \subset M$, then there exist open sets $V_1,\ldots,V_n$ such that $f^{-1}(V_i)$, $1 \leq i \leq n$, cover X and $f^{-1}(V_i)xV_i \subset M$, $1 \leq i \leq n$.

<u>Proof</u>:

For $y \varepsilon f(X)$, each $x_\alpha \varepsilon f^{-1}(y)$ has a corresponding basic open set N_α in XxY containing (x_α,y) , $N_\alpha = N(x_\alpha) \times (N_\alpha(y)$ where $N(x_\alpha)$ is open in X and $N_\alpha(y)$ is open in Y .

Now $f^{-1}(y)$ is closed, and therefore compact, so can be covered by finitely many of the $N(x_\alpha)$'s , say $N(x_1),\ldots,N(x_m)$. Let $I = \bigcap_{i=1}^{m} N_i(y)$ and $0 = \bigcup_{i=1}^{m} N(x_i)$. I and 0 are both open, and $0 \times I \subset M$. Also, $y \varepsilon I$ and $f^{-1}(y) \subset 0$. Now 0^c is compact, so $f(0^c)$ is compact and therefore closed in Y . Then $[f(0^c)]^c$ is open in Y . $y \varepsilon [f(0^c)]^c$ so $V = I \cap [f(0^c)]^c$ is an open neighborhood of y ; and $f^{-1}([f(0^c)]^c) \subset 0$ so $f^{-1}(V) \subset 0$. Thus each $y \varepsilon f(x)$ has a neighborhood V such that $[f^{-1}(V) \times V] \subset M$. For all such y , the corresponding $f^{-1}(V)$'s form an open cover of X . Then there is a finite subcover corresponding to say $V_1,\ldots,V_n$. Thus $V_1,\ldots,V_n$ are the required sets.

Theorem 5.9 The Γ_2 topology is contained in the near topology on the set of continuous functions when X is compact and Y is T_2 .

Proof:

Let $F_M = \{f \in C : G(f) \subset M \; ; \; M$ open in $XxY\}$ be a basic Γ_2 open set on C . By Lemma 5.9 , for $f \in F_M$, there are open sets $V_1,\ldots,V_n$ such that $f^{-1}(V_i)$, $1 \leq i \leq n$, cover X and $f^{-1}(V_i) \times V_i \subset M$, $1 \leq i \leq n$. Now $f(X)$ is compact and so closed in Y , so $\{[f(X)]^c , V_1,\ldots,V_n\} = \alpha$ is an open cover of Y . Then $W(f,\alpha)$ contains f and is contained in F_M since if $g \in W(f,\alpha)$, for each $x \in X$, there is an element β of α such that $f(x)$, $g(x) \in \beta$. β cannot be $[f(X)]^c$ so it is one of the V_i's . Thus $g(x) \in f^{-1}(V_i) \times V_i$ or $G(g) \subset \bigcup_{i=1}^{n} [f^{-1}(V_i) \times V_i] \subset M$. Thus F_M is open in the near topology.

Theorem 5.10 The Γ_2 topology is contained in the connected open topology on the set of continuous functions when X is compact T_2 and locally connected.

Proof:

The Γ_2 topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 by Theorem 5.2 . The compact open topology is contained in the connected open topology on the set of continuous functions when X is locally connected by Theorem 2.10. Thus $\Gamma_2 \subset T$ on C when X is compact T_2 and locally connected.

Theorem 5.11a The Γ_2 topology is contained in the τ topology

on the set of continuous functions when X is compact T_2 .

Proof:

The Γ_2 topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 by Theorem 5.2 . The compact open topology is contained in the τ topology by Theorem 2.11 . Thus $\Gamma_2 \subset \tau$ on C when X is compact T_2 .

Theorem 5.11b The Γ_2 topology is contained in the τ topology on the set of continuous functions when X is compact and Y is T_2 .

Proof:

Let $F_M = \{ f \varepsilon C : G(f) \subset M , M$ open in $XxY \}$ be a basic Γ_2 open set on C . By Lemma 5.9 there are sets $V_1,\ldots,V_n$ open in Y such that $f^{-1}(V_i)$, $1 \leq i \leq n$, cover X and $f^{-1}(V_i) \times V_i \subset M$ for $1 \leq i \leq n$. The V_i's can be taken so that $f^{-1}(V_i) \neq \phi$, so that $f \varepsilon (f,V_i)$, $1 \leq i \leq n$, and (f,V_i) , $1 \leq i \leq n$ are τ open. Thus $f \varepsilon \bigcap_{i=1}^{n} (f,V_i)$ is τ open. Let $g \varepsilon \bigcap_{i=1}^{n} (f,V_i)$. If $(x,y) \varepsilon G(g)$, x is in some $f^{-1}(V_i)$ since these sets cover X . If $x \varepsilon f^{-1}(V_j)$, then $y \varepsilon V_j$ since $g^{-1}(V_j) = f^{-1}(V_j)$ and $y = g(x)$. Thus $(x,y) \varepsilon f^{-1}(V_j) \times V_j \subset M$ so $G(g) \subset M$. Thus $f \varepsilon \bigcap_{i=1}^{n} (f,V_i) \subset F_M$ so F_M is τ-open.

Theorem 6.1 The Γ_3 topology is not contained in the pointwise convergence topology on the set of continuous functions when

$X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in the Γ_3 topology when X is T_2 by Theorem 2.6 .

The compact open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus the Γ_3 topology is not contained in the pointwise convergence topology on the continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 6.2 The Γ_3 topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 .

Proof:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The Γ_2 topology is contained in the compact open topology on the continuous functions when X is compact T_2 by Theorem 5.2 . Thus $\Gamma_3 \subset K$ or C when X is compact T_2 .

Theorem 6.3 The Γ_3 topology is contained in any σ topology on the continuous functions when X is compact T_2 .

Proof:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The Γ_2 topology is contained in any σ topology on the set of continuous functions when X is compact T_2 by Theorem 5.3 . Thus $\Gamma_3 \subset \sigma$ on C when X is compact T_2 .

Theorem 6.4 The Γ_3 topology is contained in the Γ_1 topology on the set of continuous functions when X is compact T_2 .

Proof:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The Γ_2 topology is contained in the Γ_1 topology on the set of continuous functions when X is compact T_2 by Theorem 5.4 . Thus $\Gamma_3 \subset \Gamma_1$ on C when X is compact T_2 .

Theorem 6.5 The Γ_3 topology is contained in the Γ_2 topology when X is T_1 .

Proof:

Let $U = \{U_i : 1 \leq i \leq n\}$ be a finite clover of $G(f)$, $f \varepsilon F$. Choose $p_i \varepsilon X$ such that $(p_i, f(p_i)) \varepsilon U_i$, $1 \leq i \leq n$, and let $P = \bigcup_{i=1}^{n} P_i$. For each $x \varepsilon P^c$ there exists an open set W_x containing $(x,f(x))$ and contained in $\bigcup_{i=1}^{n} U_i - (P \times Y)$. Also for each i , $1 \leq i \leq n$, there exists an open set W_{p_i} containing $(p_i, f(p_i))$ and contained in U_i . Let

$W = \bigcup_{x \varepsilon X} W_x$. Then W is a one-element clover of $G(f)$ and if $G(g) \subset W$, U is a clover of $G(g)$. So $\{f : U$ is a clover of $f\}$ is open in the Γ_2 topology.

<u>Theorem 6.7</u> The Γ_3 topology is contained in the Γ_4 topology.

<u>Proof</u>:

A finite clover of the graph of a function is clearly locally finite, so each basic Γ_3 open set is a basic Γ_4 open set.

<u>Theorem 6.8</u> The Γ_3 topology is contained in the uniform convergence topology on the set of continuous functions when X is compact T_2 .

<u>Proof</u>:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The Γ_2 topology is contained in the uniform convergence topology on the set of continuous functions when X is compact T_2 by Theorem 5.8 . Thus $\Gamma_3 \subset uc$ on C when X is compact T_2 .

<u>Theorem 6.9</u> The Γ_3 topology is contained in the near topology on the set of continuous functions when X is compact T_1 and Y is T_2 .

Proof:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Thoerem 6.5 . The Γ_2 topology is contained in the near topology on the set of continuous functions when X is compact and Y is T_2 by Theorem 5.9 . Thus $\Gamma_3 \subset N$ on C when X is compact T_1 and Y is T_2 .

Theorem 6.10 The Γ_3 topology is contained in the connected open topology on the set of continuous functions when X is compact T_2 and locally connected.

Proof:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The Γ_2 topology is contained in the connected open topology on the set of continuous functions when X is compact T_2 and locally connected by Theorem 5.10 . Thus $\Gamma_3 \subset T$ on C when X is compact T_2 and locally connected.

Theorem 6.11 The Γ_3 topology is contained in the τ topology on the set of continuous functions when X is compact and either X is T_2 or X is T_1 and Y is T_2 .

Proof:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The Γ_2 topology is contained in

the τ topology on the set of continuous functions when X is compact and either X or Y is T_2 by Theorems 5.11a or 5.11b . Thus $\Gamma_3 \subset \tau$ on C when X is compact and either X is T_2 or X is T_1 and Y is T_2 .

Theorem 7.1 The Γ_4 topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in the Γ_4 topology when X is T_2 by Theorem 2.7 . The compact open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus the Γ_4 topology is not contained in the pointwise convergence topology on C when $X = Y = [0,1]$ with the usual topology.

Theorem 7.2 The Γ_4 topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 .

Proof:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 .

The Γ_3 topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 by Theorem 6.2 . Thus $\Gamma_4 \subset K$ on C when X is compact T_2 .

<u>Theorem 7.3</u> The Γ_4 topology is contained in any σ topology on the set of continuous functions when X is compact T_2 .

<u>Proof</u>:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 . The Γ_3 topology is contained in any σ topology on the set of continuous functions when X is compact T_2 by Theorem 6.3 . Thus $\Gamma_4 \subset \sigma$ on C when X is compact T_2 .

<u>Theorem 7.4</u> The Γ_4 topology is contained in the Γ_1 topology on the set of continuous functions when X is compact T_2 .

<u>Proof</u>:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 . The Γ_3 topology is contained in the Γ_1 topology on the set of continuous functions when X is compact T_2 by Theorem 6.4 . Thus $\Gamma_4 \subset \Gamma_1$ on C when X is compact T_2 .

<u>Theorem 7.5</u> The Γ_4 topology is contained in the Γ_2

topology on the set of continuous functions when X is compact T_1 .

Proof:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 . The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . Thus $\Gamma_4 \subset \Gamma_2$ on C when X is compact T_1 .

Theorem 7.6 The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact.

Proof:

Let f be a continuous function and consider the basic Γ_4 neighborhood of f corresponding to the locally finite clover U of $G(f)$. X is compact and f continuous so $G(f)$ is compact in XxY . U is locally finite so each $(x, f(x)) \varepsilon G(f)$ has an open neighborhood that meets only a finite number of elements of U . These neighborhoods form an open cover of $G(f)$ so there is a finite subcover. Now $G(f)$ is covered by a finite number of sets each meeting only a finite number of elements of U , so $G(f)$ meets only finitely many elements of U , i.e. U is finite since $G(f)$ meets all elements of U . Thus the neighborhood corresponding to U is a Γ_3 open set.

Theorem 7.8 The Γ_4 topology is contained in the uniform

convergence topology on the set of continuous functions when X is compact T_2 .

<u>Proof</u>:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 . The Γ_3 topology is contained in the uniform convergence topology on the set of continuous functions when X is compact T_2 by Theorem 6.8 . Thus $\Gamma_4 \subset uc$ on C when X is compact T_2 .

<u>Theorem 7.9</u> The Γ_4 topology is contained in the near topology on the set of continuous functions when X is compact T_1 and Y is T_2 .

<u>Proof</u>:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 . The Γ_3 topology is contained in the near topology on the set of continuous functions when X is compact T_1 and Y is T_2 by Theorem 6.9 . Thus $\Gamma_4 \subset N$ on C when X is compact T_1 and Y is T_2 .

<u>Theorem 7.10</u> The Γ_4 topology is contained in the connected open topology on the set of continuous functions when X is compact T_2 and locally connected.

Proof:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 . The Γ_3 Topology is contained in the connected open topology on the set of continuous functions when X is compact T_2 and locally connected by Theorem 6.10 . Thus $\Gamma_4 \subset T$ on C when X is compact T_2 and locally connected.

Theorem 7.11 The Γ_4 topology is contained in the τ topology on the set of continuous functions when X is compact and either X is T_1 and Y is T_2 or X is T_2 .

Proof:

The Γ_4 topology is contained in the Γ_3 topology on the set of continuous functions when X is compact by Theorem 7.6 . The Γ_3 topology is contained in the τ topology on the set of continuous functions when X is compact and either X is T_1 and Y is T_2 or X is T_2 by Theorem 6.11 . Thus $\Gamma_4 \subset \tau$ on C when X is compact and either X is T_1 and Y is T_2 or X is T_2 .

Theorem 8.1 The uniform convergence topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in the uniform

convergence topology on the set of continuous functions by Theorem 2.8 . The compact open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus the uniform convergence topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 8.2 The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is compact.

Proof: [see Kelley [4], p. 230]

Let V be the uniformity for Y and let f be continuous. We must show that for $v \in V$ there are compact sets $A_1,\ldots,A_n$ in X and open sets $U_1,\ldots,U_n$ in Y such that $f(A_i) \subset U_i$, and if $g(A_i) \subset U_i$ for all i then $g(x) \in v[f(x)]$ for all $x \in X$.

Choose a closed symmetric member W of V such that $W \circ W \circ W \subset v$. Choose $x_1,\ldots,x_n$ such that the sets $W[f(x_i)]$ cover $f(X)$, which can be done since $f(X)$ is compact. Then let $K_i = f^{-1}(W[f(x_i)])$ and let U_i be the interior of $W \circ W[f(x_i)]$. If $g(K_i) \subset U_i$ for each i , then for each $x \in X$ there is an i such that $x \in K_i$ so $g(x) \in W \circ W[f(x_i)]$ and since $f(x) \in W[f(x_i)]$ it follows that $(g(x) , f(x)) \in W \circ W \circ W \subset v$.

<u>Theorem 8.3</u> The uniform convergence topology is contained in any σ topology on the set of continuous functions when X is compact T_2 .

<u>Proof</u>:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is compact by Theorem 8.2 . The compact open topology is contained in any σ topology when X is T_3 by Theorem 2.3 . A compact T_2 space is T_3 so $uc \subset \sigma$ on C when X is compact T_2 .

<u>Theorem 8.4</u> The uniform convergence topology is contained in the Γ_1 topology on the set of continuous functions when X is compact T_2 .

<u>Proof</u>:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is compact by Theorem 8.2 . The compact open topology is contained in the Γ_1 topology when X is T_2 by Theorem 2.4 . Thus $uc \subset \Gamma_1$ on C when X is compact T_2 .

<u>Theorem 8.5</u> The uniform convergence topology is contained in the Γ_2 topology on the set of continuous functions when X is compact T_2 .

Proof:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is compact by Theorem 8.2 . The compact open topology is contained in the Γ_2 topology when X is T_2 by Theorem 2.5 . Thus $uc \subset \Gamma_2$ on C when X is compact T_2 .

Theorem 8.6 The uniform convergence topology is contained in the Γ_3 topology on the set of continuous functions when X is compact T_2 .

Proof:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is compact by Theorem 8.2 . The compact open topology is contained in the Γ_3 topology when X is T_2 by Theorem 2.6 . Thus $uc \subset \Gamma_3$ on C when X is compact T_2 .

Theorem 8.7 The uniform convergence topology is contained in the Γ_4 topology on the set of continuous functions when X is compact T_2 .

Proof:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is

compact by Theorem 8.2 . The compact open topology is contained in the Γ_4 topology when X is T_2 by Theorem 2.7 . Thus $uc \subset \Gamma_4$ on C when X is compact T_2 .

Theorem 8.9 The uniform convergence topology is contained in the near topology on the set of continuous functions when X is compact T_2 and Y is T_2 .

Proof:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is compact by Theorem 8.2 . The compact open topology is contained in the near topology on the set of closed functions when X is T_2 by Theorem 2.9 . The continuous functions are closed if X is compact and Y is T_2 since the continuous image of a closed (implies compact) subset of X is compact (implies closed) in Y . Thus $uc \subset N$ on C if X is compact T_2 and Y is T_2 .

Theorem 8.10 The uniform convergence topology is contained in the connected open topology on the set of continuous functions when X is compact and locally connected.

Proof:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is

compact by Theorem 8.2 . The compact open topology is contained in the connected open topology on the set of continuous functions when X is locally connected by Theorem 2.10 . Thus $uc \subset T$ on C when X is compact and locally connected.

Theorem 8.11 The uniform convergence topology is contained in the τ topology on the set of continuous functions when X is compact.

Proof:

The uniform convergence topology is contained in the compact open topology on the set of continuous functions when X is compact by Theorem 8.2 . The compact open topology is contained in the τ topology by Theorem 2.11 . Thus $uc \subset \tau$ on C when X is compact.

Theorem 9.1 The near topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in the near topology on the set of closed functions (here $C \subset Q$) when X is T_2 by Theorem 2.9 . The compact open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus

the near topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 9.2 The near topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 .

Proof:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in the compact open topology on the set of continuous functions when X is compact T_2 by Theorem 5.2 . Thus $N \subset K$ on C when X is compact T_2 .

Theorem 9.3 The near topology is contained in any σ topology on the set of continuous functions when X is compact T_2 .

Proof:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in any σ topology functions when X is compact T_2 by Theorem 5.3 . Thus $N \subset \sigma$ on C when X is compact T_2 .

Theorem 9.4 The near topology is contained in the Γ_1

topology on the set of continuous functions when X is compact T_2 .

Proof:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in the Γ_1 topology on the set of continuous functions when X is compact T_2 by Theorem 5.4 . Thus $N \subset \Gamma_1$ on C when X is compact T_2 .

Theorem 9.5 The near topology is contained in the Γ_2 topology on the set of continuous functions.

Proof:

Let $W(f,\alpha)$ be a subbasic open set in the near topology, where f is continuous. For each $U \varepsilon \alpha$, $f^{-1}(U)$ is open in X , so $M = \bigcup_{U\varepsilon\alpha} f^{-1}(U) \times U$ is open in $X \times Y$. Let $g \varepsilon F_M = \{h \varepsilon C : G(h) \subset M\}$. Then for all $x \varepsilon X$, there is a $u \varepsilon \alpha$ such that $(x, g(x)) \varepsilon f^{-1}(U) \times U$. Then $f(x)$, $g(x)$ are in U so $g \varepsilon W(f,\alpha)$. Thus $f \varepsilon F_M \subset W(f,\alpha)$ so $W(f,\alpha)$ is open in the Γ_2 topology.

Theorem 9.6 The near topology is contained in the Γ_3 topology on the set of continuous functions.

Proof:

The near topology is contained in the Γ_2 topology on

the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in the Γ_3 topology by Theorem 5.6 . Thus $N \subset \Gamma_3$ on C .

<u>Theorem 9.7</u> The near topology is contained in the Γ_4 topology on the set of continuous functions.

<u>Proof</u>:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in the Γ_4 topology by Theorem 5.7 . Thus $N \subset \Gamma_4$ on C .

<u>Theorem 9.8</u> The near topology is contained in the uniform convergence topology on the set of continuous functions when X is compact T_2 .

<u>Proof</u>:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in the uniform convergence topology on the set of continuous functions when X is compact T_2 by Theorem 5.8 . Thus $N \subset uc$ on C when X is compact T_2 .

<u>Theorem 9.10</u> The near topology is contained in the

connected open topology on the set of continuous functions when X is compact T_2 and locally connected.

Proof:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in the connected open topology on the set of continuous functions when X is compact T_2 and locally connected by Theorem 5.10 . Thus $N \subset T$ on C when X is compact T_2 and locally connected.

Theorem 9.11 The near topology is contained in the τ topology on the set of continuous functions when X is compact and either X is T_2 or Y is T_2 .

Proof:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The Γ_2 topology is contained in the τ topology on the set of continuous functions when X is compact and either X is T_2 or Y is T_2 by Theorems 5.11a or 5.11b . Thus $N \subset \tau$ on C when X is compact and either X is T_2 or Y is T_2 .

Theorem 10.1 The connected open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

[illegible]

Proof:

The compact open topology is contained in the connected open topology on the set of continuous functions when X is locally connected by Theorem 2.10 . The compact open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus the connected open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ With the usual topology.

Theorem 10.2 The connected open topology is not contained in the compact open topology on the set of continuous functions when $X = Y = [0,1]$.

Proof:

The compact open topology is contained in the Γ_2 topology when X is T_2 by Theorem 2.5 . The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 10.5 . Thus the connected open topology is not contained in the compact open topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 10.3 The connected open topology is not contained in any σ topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

Any σ topology is contained in the Γ_2 topology by Theorem 3.5 . The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 10.5 . Thus the connected open topology is not contained in any σ topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 10.4 The connected open topology is not contained in the Γ_1 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The Γ_1 topology is contained in the Γ_2 topology by Theorem 4.5 . The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 10.5 . Thus the connected open topology is not contained in the Γ_1 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 10.5 The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof: (counterexample:)

Let $X = Y = [0,1]$ with the usual topology, and let

$f(x) = x$ for all $x \varepsilon X$. Consider the subbasic T-open set $W(K;U,V)$ where $U = (1/4, 3/4)$, $V = (1/2. 3/4)$, $K = (1/4, 3/4)$. For any open set M in XxY containing $G(f)$ there is an ε such that $(3/4 - 4\varepsilon, 3/4 + 4\varepsilon) \times (3/4 - 4\varepsilon, 3/4 + 4\varepsilon) \subset M$. Define

$$g(x) = \begin{cases} x & \text{if } x \varepsilon [0, 3/4 - 2\varepsilon] \\ 4x - 9/4 + 6\varepsilon & \text{if } x \varepsilon [3/4 - 2\varepsilon, 3/4 - \varepsilon] \\ 3/4 + 2\varepsilon & \text{if } x \varepsilon [3/4 - \varepsilon, 3/4 + 2\varepsilon] \\ x & \text{if } x \varepsilon [3/4 + 2\varepsilon, 1] \end{cases}$$

Then $g(x)$ is continuous and $G(g) \subset M$. But $g(x) \notin W(K;U,V)$ since $g(3/4 - \varepsilon) = 3/4 + 2\varepsilon$ so $g(K) \not\subset U \cup V = (1/4, 3/4)$. Thus $f \varepsilon W(K;U,V)$ but no basic Γ_2 neighborhood of f is contained in $W(K;U,V)$, so $W(K;U,V)$ is not open in the Γ_2 topology.

<u>Theorem 10.6</u> The connected open topology is not contained in the Γ_3 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

<u>Proof</u>:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 10.5 . Thus the connected open topology is not contained in the Γ_3 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 10.7 The connected open topology is not contained in the Γ_4 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The Γ_4 topology is contained in the Γ_2 topology on the set of continuous functions when X is compact T_1 by Theorem 7.5 . The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 10.5 . Thus the connected open topology is not contained in the Γ_4 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 10.8 The connected open topology is not contained in the uniform convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The uniform convergence topology is contained in the Γ_2 topology on the set of continuous functions when X is compact T_2 by Theorem 8.5 . The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 10.5 . Thus the connected open topology is not contained in the uniform convergence topology on the set of continuous functions when $X = Y = [0,1]$ with

the usual topology.

Theorem 10.9 The connected open topology is not contained in the near topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The connected open topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 10.5 . Thus the connected open topology is not contained in the near topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 10.11 The connected open topology is contained in the τ topology.

Proof:

Let $W(K;U,V)$ be a subbasic open set in the connected open topology, and let $f \in W(K;U,V)$. Then $f^{-1}(U) \neq \phi$ and $f^{-1}(V) \neq \phi$ so $(f,U) \cap (f,V)$ is a τ open set containing f and contained in $W(K;U,V)$. Thus $W(K;U,V)$ is open in the τ topology.

Theorem 11.1 The τ topology is not contained in the pointwise convergence topology on the set of continuous functions when

$X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in the τ topology by Theorem 2.11 . The compact open topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 2.1 . Thus the τ topology is not contained in the pointwise convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 11.2 The τ topology is not contained in the compact open topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The compact open topology is contained in the Γ_2 topology when X is T_2 by Theorem 2.5 . The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 11.5 . Thus the τ topology is not contained in the compact open topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 11.3 The τ topology is not contained in any σ topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

Any σ topology is contained in the Γ_2 topology by Theorem 3.5 . The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 11.5 . Thus the τ topology is not contained in any σ topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 11.4 The τ topology is not contained in the Γ_1 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The Γ_1 topology is contained in the Γ_2 topology by Theorem 4.5 . The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 11.5 . Thus the τ topology is not contained in the Γ_1 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 11.5 The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof: (counterexample:)

Let $X = Y = [0,1]$ with the usual topology. Let

$f(x) = x$ for all $x \varepsilon X$, and let M be any open subset of XxY containing $G(f)$. Consider the τ-open set $(f,(1/2, 1])$. Since M is open, there is an $\varepsilon > 0$ such that $(1/2 - 4\varepsilon, 1/2 + 4\varepsilon) \times (1/2 - 4\varepsilon, 1/2 + 4\varepsilon) \subset M$. Define

$$g(x) = \begin{cases} x & \text{if } x \varepsilon [1/2 + 2\varepsilon, 1] \\ 1/2 + 2\varepsilon & \text{if } x \varepsilon [1/2 - \varepsilon, 1/2 + 2\varepsilon] \\ 4x - 3/2 + 6\varepsilon & \text{if } x \varepsilon [1/2 - 2\varepsilon, 1/2 - \varepsilon] \\ x & \text{if } x \varepsilon [0, 1/2 - 2\varepsilon] \end{cases}$$

Then $g(x)$ is continuous and $G(g) \subset M$. But $g \not\varepsilon (f, (1/2, 1])$ since $g^{-1}((1/2, 1]) = (1/2 - 3/2 , 1]$ and $f^{-1}((1/2, 1]) = (1/2, 1]$. Thus $(f, (1/2, 1])$ is not Γ_2 open on the set of continuous functions.

<u>Theorem 11.6</u> The τ topology is not contained in the Γ_3 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

<u>Proof</u>:

The Γ_3 topology is contained in the Γ_2 topology when X is T_1 by Theorem 6.5 . The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 11.5 . Thus the τ topology is not contained in the Γ_3 topology on the set of

continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 11.7 The τ topology is not contained in the Γ_4 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The Γ_4 topology is contained in the Γ_2 topology on the set of continuous functions when X is compact T_1 by Theorem 7.5. The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 11.5 . Thus the τ topology is not contained in the Γ_4 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Theorem 11.8 The τ topology is not contained in the uniform convergence topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

Proof:

The uniform convergence topology is contained in the Γ_2 topology on the set of continuous functions when X is compact T_2 by Theorem 8.5 . The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 11.5 . Thus the τ topology is not contained in the uniform convergence topology on the set of continuous

functions when $X = Y = [0,1]$ with the usual topology.

<u>Theorem 11.9</u> The τ topology is not contained in the near topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

<u>Proof</u>:

The near topology is contained in the Γ_2 topology on the set of continuous functions by Theorem 9.5 . The τ topology is not contained in the Γ_2 topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology by Theorem 11.5. Thus the τ topology is not contained in the near topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

<u>Theorem 11.10</u> The τ topology is not contained in the connected open topology on the set of continuous functions when $X = Y = [0,1]$ with the usual topology.

<u>Proof</u>: (counterexample:)

Let $X = Y = [0,1]$ with the usual topology, and let $f(x) = x$ for all $x \in X$. Let $U = \bigcup_{n=1}^{\infty} (\frac{1}{2n+1}, \frac{1}{2n})$. Suppose $\bigcap_{i=1}^{m} W(K_i;U_i,V_i)$ is any basic T-open neighborhood of f . Then there is a continuous function $g \in \bigcap_{i=1}^{m} W(K_i;U_i,V_i)$ which is not in (f,U) as follows: Let x_o be the least $x \in X$ such that $x \in \overline{K_i}$ for some i for which $K_i \neq \{0\}$. If $x_o \neq 0$, there is an $n_1 > 1$

such that $\frac{1}{2n_1} < x_o$. If $x_o = 0$, rearrange subscripts so that $0 \in \overline{K_i}$ but $\{0\} \neq K_i$ for $1 \leq i \leq \ell$ and $\{0\} \neq K_i$ or $0 \notin \overline{K_i}$ for $\ell < i \leq m$. Then there is an $n_2 > 1$ such that $(0, \frac{1}{2n_2}) \subset \bigcap_{i=1}^{\ell} K_i$. Define

$$g(x) = \begin{cases} x & \text{if } x \in [\frac{1}{2n_o}, 1] \\ \frac{1}{2n_o} & \text{if } x \in [\frac{1}{2n_o + 1}, \frac{1}{2n_o}] \\ \frac{2n_o + 1}{n_o} x - \frac{1}{2n_o} & \text{if } x \in [\frac{1}{2n_o + 2}, \frac{1}{2n_o + 1}] \\ x & \text{if } x \in [0, \frac{1}{2n_o + 2}] \end{cases}$$

where $n_o > n_1$, $n_o > n_2$. Now g is continuous and $g(K_i) = f(K_i)$, $1 \leq i \leq m$ so $g \in \bigcap_{i=1}^{m} W(K_i;U_i,V_i)$ but $g^{-1}(\frac{1}{2n_o + 1/2}) \in [\frac{1}{2n_o + 2}, \frac{1}{2n_o + 1}] \not\subset f^{-1}(U)$ so $g \notin (f,U)$.

CHAPTER III

Separation Axioms

This chapter deals with the separation properties of the various function spaces defined above. The following remark will be useful in applying the theorems of Chapter II to the results obtained here.

Remark 1 If a space with topology τ_1 has the separation property T_o , T_1 or T_2 , and τ_1 is contained in a topology τ_2 , then the space with topology τ_2 also has the same separation property.

Lemma 1 If (Y, Y) is T_o then $(F, p.c.)$ is T_o .

Proof:

Suppose f , $g \in F$, $f \neq g$. Then there is an $x \in X$ such that $f(x) = y_1 \neq y_2 = g(x)$. Then y_1 or y_2 , say y_1 , has an open neighborhood U not containing the other point, y_2 . Then (x,U) is an open neighborhood of f not containing g so $(F,p.c.)$ is T_o .

Lemma 2 If (Y, Y) is T_1 then $(F,p.c.)$ is T_1 .

Proof:

Suppose f , $g \in F$, $f \neq g$. Then there exists an $x \in X$

such that $f(x) = y_1 \neq y_2 = g(x)$. y_1 has an open neighborhood U that does not contain y_2 so (x,U) is an open neighborhood of f that does not contain g . Thus $(F,p.c.)$ is T_1 .

<u>Lemma 3</u> If $(Y, Ⓨ)$ is T_2 then $(F,p.c.)$ is T_2 .

<u>Proof</u>:

Suppose f , $g \varepsilon F$, $f \neq g$. Then there exists an $x \varepsilon X$ such that $f(x) = y_1 \neq y_2 = g(x)$. Then there are open neighborhoods U_1 and U_2 of y_1 and y_2 respectively such that $U_1 \cap U_2 = \phi$. Then (x,U_1) and (x,U_2) are disjoint open neighborhoods of f and g respectively, so $(F,p.c.)$ is T_2 .

<u>Lemma 4</u> In the τ topology basic open neighborhoods of constant functions are of the form $\tau(X,U)$, in the sense that finite intersections of the defining subbasic open neighborhoods of constant functions are of that form.

<u>Proof</u>:

A subbasic open enighborhood of a constant function is of the form $\tau(f^{-1}(U), U)$ where f is constant and $f^{-1}(U) \neq \phi$. But then $f^{-1}(U) = X$. Now $\bigcap_{i=1}^{n} (X,U_i) = \tau(X, \bigcap_{i=1}^{n} U_i)$ is of the required form.

<u>Lemma 5</u> If (F,τ) is T_o then $(Y, Ⓨ)$ is T_o .

Proof:

If (Y, Y) is not T_o there are points y_1 , $y_2 \varepsilon Y$ such that every open neighborhood of either contains the other. But then every basic open neighborhood of y_1^* , $\tau(X,U)$, contains y_2^* since if $y_1 \varepsilon U$, $y_2 \varepsilon U$. Similarly $y_2^* \varepsilon \tau(X,U)$ implies $y_1^* \varepsilon \tau(X,U)$ so (F,τ) is not T_o .

Lemma 6 If (F,τ) is T_1 then (Y, Y) is T_1 .

Proof:

If (Y, Y) is not T_1 then there are points y_1 , $y_2 \varepsilon Y$ such that every open neighborhood of y_1 contains y_2 . But then every basic open neighborhood of y_1^* , $\tau(X,U)$, contains y_2^* since $y_1 \varepsilon U$ implies $y_2 \varepsilon U$. Thus (F,τ) is not T_1 .

Lemma 7 If (F,τ) is T_2 , then (Y, Y) is T_2 .

Proof:

If (Y, Y) is not T_2 there are points y_1 , $y_2 \varepsilon Y$ such that every open neighborhood of y_1 meets every open neighborhood of y_2 . Then for the functions y_1^* and y_2^* , every basic τ open neighborhood $\tau(X,U)$ of y_1^* and $\tau(X,V)$ of y_2^* there is a $y_3 \varepsilon U \cap V$. Then $y_3^* \varepsilon \tau(X,U) \cap \tau(X,V)$ so (F,τ) is not T_2 .

Theorem 1 The pointwise convergence, compact open, connected

open and τ topologies satisfy axiom T_i if and only if (Y, Y) satisfies axiom T_i , $i = 0,1,2$.

Proof:

If (Y, Y) is T_i , $i = 0,1,2$, then the pointwise convergence topology also has that property by Lemmas 1, 2, 3 . The pointwise convergence topology is contained in the compact open topology by Theorem 1.2 , in the connected open topology by Theorem 1.10 and in the τ topology by Theorem 1.11 . Thus by Remark 1 , these topologies also satisfy axiom T_i .

If (F,p.c.) , (F,k) , (F,T) or (F,τ) satisfies axiom T_i , $i = 0,1,2$, then since the τ topology contains the pointwise convergence topology by Theorem 1.11 , the compact open topology by Theorem 2.11 , and the connected open topology by Theorem 10.11 , Remark 1 implies that (F,τ) satisfies that axiom. Then for T_o , T_1 or T_2 , Lemma 5, 6 or 7 respectively, implies that (Y, Y) also satisfies that axiom.

Lemma 8 If (F,Γ_4) is T_o then (Y, Y) is T_o .

Proof:

If (Y, Y) is not T_o , there are points y_1 , $y_2 \varepsilon Y$ such that every open neighborhood of y_1 contains y_2 and vice versa. Let $x \varepsilon X$. Then for any locally finite clover $\mathcal{U}$ of $G(y_1^*)$, (x,y_1) has an open neighborhood U meeting only a finite number of elements of

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$\underset{\sim}{U}$. Let U_1 be the intersection of U with all clover elements containing (x,y_1) . U_1 is open so there are open sets $U_x \subset X$ and $U_y \subset Y$ such that $(x,y_1) \varepsilon U_x \times U_y \subset U_1$. U_y contains y , so it contains y_2 so $(x,y_2) \varepsilon U_1$. Thus $\underset{\sim}{U}$ is locally finite at (x,y_2) since $(x,y_2) \varepsilon U_1 \subset U$, and U covers (x,y_2) . (x,y_2) meets every clover element that (x,y_1) meets, so $\underset{\sim}{U}$ is a locally finite clover of $G(y_2^*)$ since x was arbitrary. Similarly every locally finite clover of $G(y_2^*)$ is a locally finite clover of $G(y_1^*)$ so (F,Γ_4) is not T_o .

<u>Lemma 9</u> If (F,Γ_4) is T_1 then $(Y, \textcircled{Y})$ is T_1 .

<u>Proof</u>:

If $(Y, \textcircled{Y})$ is not T_1 there are points y_1 , $y_2 \varepsilon Y$ such that every open neighborhood of y_1 contains y_2 . Let $x \varepsilon X$. Then for any locally finite clover $\underset{\sim}{U}$ of $G(y_1^*)$, (x,y_1) has an open neighborhood U meeting only a finite number of elements of $\underset{\sim}{U}$. Let U_1 be the intersection of U with all clover elements containing (x,y_1) . U_1 is open so there are open sets $U_x \subset X$ and $U_y \subset Y$ such that $(x,y_1) \varepsilon U_x \times U_y \subset U_1$. U_y contains y_1 so it contains y_2 so $(x,y_2) \varepsilon U_1$. Thus $\underset{\sim}{U}$ is locally finite at (x,y_2) since $(x,y_2) \varepsilon U_1 \subset U$ and $\underset{\sim}{U}$ covers (x,y_2) . (x,y_2) meets every clover element that (x,y_1) meets, so $\underset{\sim}{U}$ is a locally finite clover of $G(y_2^*)$ since x was arbitrary. Thus every locally finite clover of $G(y_1^*)$ is a locally finite clover of $G(y_2^*)$ so (F,Γ_4) is not T_1 .

<u>Lemma 10</u> If (F,Γ_4) is T_2 then $(Y,Ⓨ)$ is T_2 .

<u>Proof</u>:

If $(Y,Ⓨ)$ is not T_2 there are points y_1 , $y_2 \varepsilon Y$ such that every open neighborhood of y_1 meets every open neighborhood of y_2 . Let $x \varepsilon X$. Then for any locally finite clover $\underset{\sim}{U}^1$ of $G(y_1^*)$, (x,y_1) has an open neighborhood U^1 meeting only a finite number of elements of $\underset{\sim}{U}^1$. Let U_1^1 be the intersection of U^1 with all clover elements containing (x,y_1) . U_1^1 is open so there are open sets $U_y^1 \subset Y$ and $U_x^1 \subset X$ such that $(x,y_1) \varepsilon U_x^1 \times U_y^1 \subset U_1^1$. Similarly for any locally finite clover U^2 of $G(y_2^*)$ there are corresponding sets U^2, U_1^2, U_x^2 and U_y^2 , corresponding to the point (x,y_2) . Now $y_1 \varepsilon U_y^1$ and $y_2 \varepsilon U_y^2$ so there is a $y_3 \varepsilon U_y^1 \cap U_y^2$. Then $\underset{\sim}{U}^1$ and $\underset{\sim}{U}^2$ are locally finite at (x,y_3) since $(x,y_3) \varepsilon U_1^1 \cap U_1^2 \subset U^1 \cap U^2$. Also, $\underset{\sim}{U}^1$ and $\underset{\sim}{U}^2$ cover (x,y_3) , and (x,y_3) meets every $\underset{\sim}{U}^1$ element that (x,y_1) meets, and every $\underset{\sim}{U}^2$ element that (x,y_2) meets.
Thus $\underset{\sim}{U}^1$ and $\underset{\sim}{U}^2$ are locally finite clovers of the graph of the function defined as $f(x) = y_3$. This function is defined as above for each $x \varepsilon X$. Thus the basic Γ_4 open sets corresponding to $\underset{\sim}{U}^1$ and $\underset{\sim}{U}^2$ have a non empty intersection, so (F,Γ_4) is not T_2 .

<u>Theorem 2</u> If X is T_1 then any σ topology, the Γ_1 topology, the Γ_2 topology, the Γ_3 topology and the Γ_4 topology are each T_i if and only if $(Y,Ⓨ)$ is T_i , $i = 0,1,2$.

Proof:

If $(Y, \textcircled{Y})$ is T_i , the pointwise convergence topology is T_i by Lemma 1 for $i = 0$, Lemma 2 for $i = 1$ or Lemma 3 for $i = 2$. If X is T_1 , the pointwise convergence topology is contained in any σ topology by Theorem 1.3 , in the Γ_1 topology by Theorem 1.4 , in the Γ_2 topology by Theorem 1.5 , in the Γ_3 topology by Theorem 1.6 and in the Γ_4 topology by Theorem 1.7 . Thus by Remark 1 , all these topologies satisfy axiom T_i .

If (F,σ) , (F,Γ_1) , (F,Γ_2) , (F,Γ_3) or (F,Γ_4) is T_i then (F,Γ_4) is T_i by Remark 1 and the fact that the Γ_4 topology contains any σ topology by Theorem 3.7 , the Γ_1 topology by Theorem 4.7 , the Γ_2 topology by Theorem 5.7 and the Γ_3 topology by Theorem 6.7 . But (F,Γ_4) being T_i implies that $(Y, \textcircled{Y})$ is T_i by Lemma 8 for $i = 0$, Lemma 9 for $i = 1$ or Lemma 10 for $i = 2$.

Theorem 3 The near topology is T_1 if and only if $(Y, \textcircled{Y})$ is T_1 .

Proof:

If $(Y, \textcircled{Y})$ is T_1 the pointwise convergence topology is T_1 by Lemma 2 . Since the pointwise convergence topology is contained in the near topology when Y is T_1 by Theorem 1.9 , Remark 1 shows that (F,N) is T_1 .

If (F,N) is T_1 , let y_1 , $y_2 \in Y$, $y_1 \neq y_2$. Let $x \in X$. Let f , $g \in F$ such that $f = g$ on $X - x$ and $f(x) = y_1$, $g(x) = y_2$.

Then $f \neq g$ so there is an open neighborhood $W(h,\alpha)$ of f with $g \notin W(h,\alpha)$, where $h \in F$, α an open cover of Y. Then $f(x) \in \alpha^* h(x)$ and $g(x) \notin \alpha^* h(x)$, so $y_2 = g(x) \notin U$ where U is the member of α containing $y_1 = f(x)$ and $h(x)$. So $(Y, \textcircled{Y})$ is T_1.

Theorem 4 The near topology is T_2 if and only if $(Y, \textcircled{Y})$ is T_2.

Proof:

If $(Y, \textcircled{Y})$ is T_2, the pointwise convergence topology is T_2 by Lemma 3. Since the pointwise convergence topology is contained in the near topology when Y is T_1 by Theorem 1.9, Remark 1 shows that (F,N) is T_2.

If (F,N) is T_2, let $y_1 \neq y_2 \in Y$. Consider y_1^* and y_2^*. Then there are disjoint open neighborhoods $\bigcap_{i=1}^{n} W(f_i,\alpha_i)$ of y_1^* and $\bigcap_{j=1}^{m} W(g_j,\alpha_j)$ of y_2^* where f_i, $g_j \in F$, α_i, β_j are open covers of Y. For $x \in X$ let $U_{x,i} \in \alpha_i$ and $V_{x,j} \in \alpha_j$ be such that y_1, $f_i(x) \in U_{x,i}$ and y_2, $g_j(x) \in V_{x,j}$. Then $\bigcap_{i=1}^{n} U_{x,i}$ and $\bigcap_{j=1}^{n} V_{k,j}$ are disjoint open neighborhoods of y_1 and y_2 respectively. If not, any function $h \in F$ such that $h(x) \in (\bigcap_{i=1}^{n} U_{k,i}) \cap (\bigcap_{j=1}^{m} V_{k,j})$ will be in $\bigcap_{i=1}^{n} W(f_i,\alpha_i) \cap \bigcap_{j=1}^{m} W(g_j,\alpha_j)$; which is empty. Thus $(Y, \textcircled{Y})$ is T_2.

Theorem 5 If $(Y, \textcircled{Y})$ is regular then $(F,\text{p.c.})$ is regular.

Proof:

It suffices to consider only subbasic sets. Let $f \varepsilon (x,U)$. Then there is an open set V in Y such that $f(x) \varepsilon V$, $\overline{V} \subset U$. Now $f \varepsilon (x,V)$ and $\overline{(x,V)} \subset (x,\overline{V}^c)^c \subset (x,U)$ so $(F,p.c.)$ is regular.

Theorem 6 If $(Y, Ⓨ)$ is regular then (C,K) is regular.

Proof:

It suffices to consider only subbasic sets. Let $f \varepsilon (A,U)$, A compact and f continuous. Then $f(A)$ is compact, and there exists an open set $V \subset Y$ such that $f(A) \subset V \subset \overline{V} \subset U$. Now $f \varepsilon (A,V)$ and $\overline{(A,V)} \subset (A,\overline{V}) \subset (A,U)$ if $(A,\overline{V})$ is closed. [$\overline{V}$ is not necessarily an open set as in the definition of the (A,U) notation, but the meaning here is $(A,\overline{V}) = \{f \varepsilon C : f(A) \subset \overline{V}\}$.] $(A,\overline{V})$ is closed since $(A,\overline{V}) = \bigcap_{x \varepsilon A} (X,\overline{V}^c)^c$. Thus (C,K) is regular.

Theorem 7 If $(Y, Ⓨ)$ is completely regular then $(F,u.c.)$ is completely regular.

Proof:

A topological space is completely regular if and only if its topology is the topology of some uniformity for the space. Thus if $(Y, Ⓨ)$ is completely regular it has a uniformity which yields a uniform convergence topology for F . But the uniform convergence topology is defined as the topology of a uniformity for F , so

(F,u.c.) is completely regular.

<u>Theorem 8</u> If X is compact T_2 then the following topologies are equivalent on the set of continuous functions:

1) the compact open topology
2) any σ topology
3) the Γ_1 topology
4) the Γ_2 topology
5) the Γ_3 topology
6) the Γ_4 topology

If Y is a uniform space:

7) the uniform convergence topology

If Y is T_2 :

8) the near topology.

<u>Proof</u>:

the compact open topology is contained in any σ topology if X is T_3 , which is implied by compact T_2 , by Theorem 2.3 . Any σ topology is contained in the Γ_1 topology by Theorem 3.4 . The Γ_1 topology is contained in the Γ_2 topology by Theorem 4.5 . The Γ_2 topology is contained in the Γ_3 topology by Theorem 5.6 . The Γ_3 topology is contained in the Γ_4 topology by Theorem 6.7 . And the Γ_4 topology is contained in the compact

open topology on the set of continuous functions when X is compact T_2 by Theorem 7.2 .

If Y is a uniform space, the uniform convergence topology is defined and equivalent to the compact open topology on the set of continuous functions when X is compact by Theorems 2.8 and 8.2 .

If Y is T_2 , the near topology is equivalent to the Γ_2 topology on the set of continuous functions when X is compact by Theorems 5.9 and 9.5 .

<u>Corollary 1</u> If (Y, Y) is completely regular and X is compact T_2 then on the continuous functions the compact open topology, any σ topology, the Γ_i , $i = 1,2,3,4$ topology, and the uniform convergence topology are completely regular.

<u>Proof</u>:

By Theorem 7 the uniform convergence topology is completely regular, and by Theorem 8 , the compact open, any σ , and the Γ_i , $i = 1,2,3,4$ topology are all equivalent, and so also completely regular.

<u>Theorem 9</u> If Y is a metric space then Y has a uniformity for which (F,u.c.) is a metric space with the metric

$d^*(f,g) = \sup_{x \in X} d(f(x), g(x))$ where d is the metric for Y .

Proof:

If Y is a metric space it has a uniformity U generated by sets of the form $V_r = \{(y_1,y_2) : y_1 , y_2 \in Y , d(y_1,y_2) < r\}$ for each $r > 0$. Then the uniform convergence topology is defined from a uniformity generated by sets of the form $W(V_r) = \{(f,g) : (f(x),g(x)) \in V_r$ for each $x \in X\}$ for $V \in U$. This gives a neighborhood system at the point f generated by sets of the form $\{g \in F : (f,g) \in W(V_r)\}$ i.e. $\{g \in F : d(f(x), g(x)) < r$ for all $x \in X\}$ which is a neighborhood base for the required metric d^*.

Theorem 10 If Y is a metric space and X is compact T_2 then on the set of continuous functions the following topologies are equivalent:

1) the metric topology with metric

$$d^*(f,g) = \sup_{x \in X} d(f(x), g(x)) , \quad d \text{ the metric of } Y .$$

2) the compact open topology

3) any σ topology

4) the Γ_1 topology

5) the Γ_2 topology

6) the Γ_3 topology

7) the Γ_4 topology

8) the uniform convergence topology

9) the near topology

Proof:

Since Y is metric, it has a uniformity, and is T_2.

Thus by Theorem 8 , since X is compact T_2 , topologies 2) through 9) are equivalent on the set of continuous functions. The metric topology 1) is equivalent to the uniform convergence topology 8) by Theorem 9 , so all topologies listed are equivalent.

<u>Lemma 11</u> For any subset A of $X x Y$ the set of non constant functions whose graphs meet A is open in the τ topology if (Y, Y) is T_1 .

<u>Proof</u>:

Let f be non constant, and let $G(f)$ meet A . Let $(x_o, y_o) \varepsilon G(f) \cap A$. Then (f, y_o^c) is a τ-open set containing only non constant functions whose graphs contain the point $(x_o, y_o) \varepsilon A$. [y_o^c is open since (Y, Y) is T_1 and $f^{-1}(y_o^c) \neq \phi$ since f is not constant.]

<u>Lemma 12</u> If Y is T_1 , the τ-closure of $\tau(P,U)$ is $\tau(P,U) \cup [\cup y_\alpha^*]$ where $y_\alpha \varepsilon \partial U$.

<u>Proof</u>:

Let $A = [P^c \times U] \cup [P \times U^c]$. By Lemma 11 , the set of non constant functions whose graphs meet A is open. Then the complement of this set, $\tau(P,U) \cup C$, is closed. So $\partial\tau(P,U) \subset C$.

If $y_o \notin \partial U$, either $y_o \varepsilon \overline{U}^c$ or $y_o \varepsilon U$. If $y_o \varepsilon \overline{U}^c$, $\tau(X, \overline{U}^c)$ is an open neighborhood of y_o^* that misses $\tau(P,U)$. If

$y_o \varepsilon U$, (X,U) is an open neighborhood of y_o^* that misses $\tau(P,U)$ unless $P = X$. In this case $y_o^* \varepsilon \tau(P,U) = \tau(X,U)$. Thus in any case y_o^* is not a boundary point of $\tau(P,U)$.

If $Y_o \varepsilon \partial U$, every basic (by Lemma 4) neighborhood $\tau(X,V)$ of y_o^* contains a function

$$g(x) = \begin{cases} y & \text{for } x \varepsilon P^c \text{ where } y^1 \varepsilon U \cap V \\ y^1 & \text{for } x \varepsilon P . \end{cases}$$

$U \cap V \neq \phi$ since $y_o \varepsilon V$, $y_o \varepsilon \partial U$ and $V \varepsilon \textcircled{Y}$. Now $g \varepsilon \tau(P,U)$, so $y_o^* \varepsilon \partial\tau(P,U)$. Thus $\overline{\tau(P,U)} = \tau(P,U) \cup [\cup\, y_\alpha^*]$ where $y_\alpha \varepsilon \partial U$.

<u>Lemma 13</u> Suppose $(Y, \textcircled{Y})$ is T_1. If $[\tau(P,U) \cup \tau(Q,V)] \cap C = \phi$ and $\partial U \cap \partial V = \phi$ then $[\tau(P,U) \cap \tau(Q,V)] \cap [\bigcap_{i=1}^{n} \tau(P_i,U_i)]$ is both open and closed.

<u>Proof</u>:

Let $I = \tau(P,U) \cap \tau(Q,V)$.

By Lemma 12

$$\overline{\tau(P,U)} = \tau(P,U) \cup \partial^* U$$
$$\overline{\tau(Q,V)} = \tau(Q,V) \cup \partial^* V .$$
$$\overline{I} \subset \overline{\tau(P,U)} \cap \overline{\tau(Q,V)}$$

But $\tau(P,U) \cap \partial^* V = \tau(Q,V) \cap \partial^* U = \partial^* U \cap \partial^* V = \phi$ by hypothesis, so

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$\overline{I} \subset I$. By definition $I \subset \overline{I}$ so $I = \overline{I}$ and I is closed. Also, $I \cap C = \phi$.

$$\overline{I \cap [\bigcap_{i=1}^{n} \tau(P_i,U_i)]} \subset \overline{I} \cap [\bigcap_{i=1}^{n} \overline{\tau(P_i,U_i)}]$$

$$= I \cap [\bigcap_{i=1}^{n} (\tau(P_i,U_i) \cup \partial^* U_i)]$$

$$= \{I \cap [\bigcap_{i=1}^{n} \tau(P_i,U_i)]\} \cup \{\bigcup_{i=1}^{n} \bigcup_{j \neq i} (\tau(P_i,U_i) \cap \partial^* U_j \cap I)\} \cup \{\bigcap_{i=1}^{n} (\partial^* U_i \cap I)\}$$

Now $I \cap C = \phi$ so

$$\overline{I \cap [\bigcap_{i=1}^{n} (P_i,U_i)]} \subset I \cap [\bigcap_{i=1}^{n} \tau(P_i,U_i)]$$

so as before $I \cap [\bigcap_{i=1}^{n} \tau(P_i,U_i)]$ is closed. It is open since it is the finite intersection of open sets.

Theorem 11 (Y, Y) is T_3 if and only if (F,τ) is T_3 .

Proof:

If (Y, Y) is T_3 , let $A = \bigcap_{i=1}^{n} \tau(P_i,U_i)$ be a basic τ-open set containing f .

Case 1 f is not constant.

If f is not constant, it takes at least two values y_1 and y_2 . Then (f,y_1^c) and (f,y_2^c) are open and contain no constant functions

since $f^{-1}(y_1^c)$, $f^{-1}(y_2^c)$ are neither ϕ nor X . Also, $\partial y_1^c \subset y_1 \neq y_2 \supset \partial y_2^c$ so $\partial y_1^c \cap \partial y_2^c = \phi$. Thus by Lemma 13 , $B = A \cap (f, y_1^c) \cap (f, y_2^c)$ is both open and closed. Then $f \in B = \overline{B} \subset A$.

Case 2 f is constant.

Since f is constant, $P_i = X$, $1 \leq i \leq n$. Let $V = \bigcap_{i=1}^{n} U_i$. Then $A = \tau(X,V)$. Since Y is T_3 there is a $W \in (Y)$ such that, if $f = Y^*$ and since $y \in V$, $y \in W$, $\overline{W} \subset V$.

Now $\overline{\tau(X,W)} = \tau(X,W) \cup \partial^* W$ by Lemma 12 . $\tau(X,W) \subset \tau(X,V)$ since $W \subset V$, and $\partial^* W \subset \tau(X,V)$ since $\overline{W} \subset V$. Thus $\overline{\tau(X,W)} \subset \tau(X,V) = A$. So $f = y^* \in \tau(X,W)$ and $\overline{\tau(X,W)} \subset \tau(X,V) = A$.

(f,τ) is T_1 since $(Y, (Y))$ is T_1 by Theorem 1 . Thus (F,τ) is T_3 .

If (F,τ) is T_3 , then $(Y, (Y))$ is T_1 since (F,τ) is T_1 by Theorem 1 .

If $y \in U \in (Y)$ consider $y^* \in \tau(X,U)$. There is a basic open set $\tau(X,V)$, by Lemma 4 , containing y^* such that $\overline{\tau(X,V)} \subset \tau(X,U)$ since (F,τ) is T_3 . But then $y \in V$ and since $\overline{\tau(X,U)} = \tau(X,U) \cup \partial^* U$ by Lemma 12 we have $V \subset U$. Thus $(Y, (Y))$ is T_3 .

<u>Theorem 12</u> $(Y, (Y))$ is a Tychanoff space if and only if

(F,τ) is a Tychanoff space.

<u>Proof</u>:

If (Y, Y) is a Tychanoff space let $A = \bigcap_{i=1}^{n} \tau(P_i, U_i)$ be a basic open set containing f .

Case 1 f is not constant.

If f is not constant it takes at least two values y_1 and y_2 . Then (f, y_1^c) and (f, y_2^c) are open and contain no constant functions since $f^{-1}(y_1^c)$, $f^{-1}(y_2^c)$ are neither X nor ϕ . Also $\partial y_1^c \subset y_1 \neq y_2 \supset \partial y_2^c$ so $\partial y_1^c \cap \partial y_2^c = \phi$. Thus by Lemma 13 , $B = A \cap (f, y_1^c) \cap (f, y_2^c)$ is both open and closed. Then the function H defined so that $H(B) = 0$ and $H(B^c) = 1$ is continuous into $[0,1]$ and maps f to 0 , A^c to 1 .

Case 2 f is constant.

Since f is constant, $P_i = X$, $1 \leq i \leq n$. Let $f = y^*$, $V = \bigcap_{i=1}^{n} U_i$. Then $A = \tau(X,V)$. Since (Y, Y) is a Tychanoff space, there is a continuous function g mapping Y into $[0,1]$ such that $g(y) = 0$, $g(y^c) = 1$. For some $x_o \in X$ define $H(h) = g \circ h(x_o)$ if $h \in \tau(X,V)$ and $H(h) = 1$ if $h \notin \tau(X,V)$. H clearly maps f to 0 and A^c to 1 . To show that H is continuous let W be open in $[0,1]$. If $1 \notin W$, $H^{-1}(W) = \{h : h^{-1}(x_o) \in g^{-1}(W)\} \cap A$. Since g is continuous, $g^{-1}(W)$ is open. Now $\{h : h(x_o) \in g^{-1}(W)\}$ is open in the pointwise

convergence topology, which is contained in the τ topology by Theorem 1.11 . Thus it is open in the τ topology. So $H^{-1}(W)$ is open since A is open. If $1 \varepsilon W$, $H^{-1}(W) = [\{h : h(x_o) \varepsilon g^{-1}(W)\} \cap A] \cup A^c$ Now $\{h : h(x_o) \varepsilon g^{-1}(W)\} \cap A$ is open as above, and the interior of A^c is open so it must be shown that ∂A^c is interior to $H^{-1}(W)$. $\partial A^c = \partial A = \{y^* : y \varepsilon \partial V\}$ by Lemma 12 . Now $g^{-1}(W)$ is open and since $1 \varepsilon W$, $V^c \subset g^{-1}(W)$. That is, $\partial V \subset g^{-1}(W)$. Then $\partial A \subset \tau(X,g^{-1}(W))$ which is open in (F,τ) . But $H(\tau(X,g^{-1}(W)) \cap A) \subset W$ and $H(\tau(X,g^{-1}(W)) \cap A^c) \subset W$ so $H(\tau(X,g^{-1}(W))) \subset W$ so $\tau(x,g^{-1}(W))) \subset H^{-1}(W)$. Thus ∂A^c is interior to $H^{-1}(W)$ and H is continuous.

Since (Y, Y) is T_1 , (F,τ) is T_1 by Theorem 1 .

If (F,τ) is a Tychanoff space, let $y_o \varepsilon U \varepsilon Y$. Then there is a continuous function f mapping F into $[0,1]$ such that $f(y_o^*) = 0$ and $f(\tau(X,U)^c) = 1$. Define $H(Y) = f(y^*)$ for all $y \varepsilon Y$. Then clearly H maps y_o to 0 and U^c to 1 and Y into $[0,1]$. To show H is continuous, let V be open in $[0,1]$. Let $y \varepsilon H^{-1}(V)$. Then $y^* \varepsilon f^{-1}(V)$. Then some basic open set $\tau(X,W)$, by Lemma 4 ; containing y^* is contained in $f^{-1}(V)$ which is open since f is continuous. But then W is an open set containing Y and contained in $H^{-1}(V)$, so $H^{-1}(V)$ is open. So H is continuous.

Since (F,τ) is T_1 , (Y, Y) is T_1 by Theorem 1 .

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B29906